3. AXIOMS OF SET THEORY

§3.1. Axioms

The modern approach to mathematics is to set up a system of axioms for a particular branch of mathematics and then to prove theorems only using these axioms as a foundation. This takes out the need for intuition. The concepts involved in that subject are left undefined.

In many ways a set of axioms is like a mathematical creed. You can't prove anything from nothing – in mathematics just as in religion. You have to begin with statements that might seem reasonable, such as "There is a God who communicates with humans" or "given a line, and a point not on that line, there exists a unique line through that point parallel to the given line". Religious people are often ridiculed for believing things they can't prove. But mathematicians must do the same.

The Euclidean axioms were considered to be intuitively obvious, and therefore true, but in the nineteenth century certain mathematicians decided to vary one of these axioms and so created non-Euclidean geometries. Then, in the twentieth century, cosmologists decided that, although Euclidean geometry was sufficiently accurate on a small scale, a non-Euclidean geometry was needed to describe the vast distances of the universe.

It can be shown that the Euclidean axioms describe just one geometry. However most sets of axioms can describe many different objects. The axioms for a vector space over a field, for example, describe many different vector spaces. However, for finite-dimensional vector spaces there is essentially only one vector space of each dimension. In the case of groups there's a much greater variety of examples. We can no longer think of the axioms as self-evident. In effect the set of axioms are just part of the definition of a vector space, or a group.

Now I'm going to attempt to devise a set of axioms for the *whole* of mathematics. On the basis of these axioms one can potentially prove everything that is known about mathematics! Well, this book is not big enough to encompass all mathematical knowledge. Rather it will go far enough for it to be obvious that this mammoth task is possible.

Actually, we'll be considering axioms for set theory. The only objects in our theory will be sets. You can think of a set as a collection of 'things' and there don't need to be any common properties that these 'things' have in common, like a dinner set. As Lewis Carroll said, they can be as disparate as "shoes and ships and sealing wax – of cabbages and kings". Lewis Carroll was a mathematician and he was making an explicit reference here to set theory.

Now mathematics is bubbling with all sorts of 'things' – numbers and functions and matrices and curves to name just a few. These don't appear to be sets but we're going to define them as sets. However, before we begin that journey and consider the axioms for set theory, let's think further about axioms.

§3.2. Models

A **model** for a set of axioms is simply an object that satisfies those axioms. So a model for group theory will be any one of the many examples of a group. In the case of the Euclidean axioms it can be shown that there's essentially only one model, but for most axiomatic systems there are many. However a set of axioms may have no models at all! We say that such a set of axioms is **inconsistent**.

We could take an inconsistent set of axioms and prove lots of theorems, but that would be a waste of time if there was nothing that satisfied those axioms.

Proving that a set of axioms is **consistent** can be very easy. We simply come up with a model. For example, the group axioms are consistent because the set {1} is a group.

But how can we prove that a set of axioms is inconsistent? We use those axioms to obtain a contradiction. I'll illustrate these concepts by a couple of baby examples of sets of axioms. They're of no mathematical significance whatsoever – they're just for training purposes.

I'll provide two, alternative, sets of axioms for a thing called a 'slice'. Now you know what a slice is in ordinary life, but you have no idea what a slice is in mathematics. That's because I just made it up as an undefined object for the purpose of this demonstration. So rid your mind of properties of physical slices, such as thinness and flatness. A slice here is an undefined object that has an undefined operation called 'division'. Again this has not very much to do with division as you know it in mathematics, although you may notice some similarities. A slice is a set where a/b is defined for all a, b in the set. The only thing we're allowed to know about this binary operation are the axioms. So, empty your mind of anything that the words 'slice' and 'division' suggest and treat them as being undefined.

A **slice** is a set S, together with an operation a/b such that:
(1) $a/b \in$ S for all $a, b \in$ S;
(2) there exists $0 \in$ S such that $0/x = 0$ for all $x \in$ S;
(3) there exists $1 \in$ S such that $1 \neq 0$ and
$x/1 = x$ for all $x \in$ S;
(4) If $a/b = c$ then $a/c = b$ for all $a, b, c \in$ S.

Theorem 1: There are no slices. That is, these axioms are inconsistent.
Proof: $0/1 = 0$ by Axioms (3) and (2).
Hence $0/0 = 1$ by Axiom (3).
But $0/0 = 0$ by Axiom (2).

Let's modify the definition of a slice as follows.

A **strawberry slice** is a set S, together with an operation a/b such that:

(1) $a/b \in$ S for all a, $b \in$ S;

(2) there exists $0 \in$ S such that $0/x = 0$ for all $x \in$ S;

(3) there exists $1 \in$S such that $0 \neq 1$ and $x/1 = x$ for all $x \in$ S;

(4) $a/(b/c) = c/(b/\mathrm{a})$ for all a, b, $c \in$ S.

Example 1: An example of a slice is S = {0, 1} with:

$0/0 = 0$;
$0/1 = 0$;
$1/0 = 0$;
$1/1 = 1$.

Note that in S, $a/(b/c) = 0$ in all cases except where:

$$a = b = c = 1.$$

In that case $c/(b/\mathrm{a}) = 1$.

So S is a model for strawberry slices and so the axioms for a strawberry slice are consistent. Let's prove a few theorems about strawberry slices.

Theorem 2: In a strawberry slice, S:

$$(a/b)/c = (a/c)/b \text{ for all } a, \mathrm{b}, \mathrm{c} \in \mathrm{S}.$$

Proof: $(a/b)/c = (a/b)/(c/1)$ by (2)
$= 1/(c/(a/b)$ by (4)
$= 1/(b/(a/c))$ by (4)
$= (a/c)/(b/1)$ by (4)
$= (a/c)/b$ by (2).

Define $\infty = 0/1$.

Theorem 3: In a strawberry slice, $x/\infty = 0$ for all x.
Proof: $x/\infty = x/(1/0)$ by definition of ∞
$= 0/(1/x)$ by (4)
$= 0$ by (2)

Theorem 4: In a strawberry slice $x/0 = \infty$ for all x.
Proof: $x/0 = x/(0/1)$ by (3)
$= 1/(0/x)$ by (4)
$= 1/0$ by (2) $= \infty$.

We seem to be developing quite a theory here. But things are about to collapse a little bit.

Theorem 5: $0 = \infty$.
Proof: $\infty = 1/0$ by definition
$= 1/(\infty/\infty)$ by Theorem 3
$= \infty/(\infty/1)$ by (4)
$= \infty/\infty$ by (3)
$= 0$ by Theorem 3.

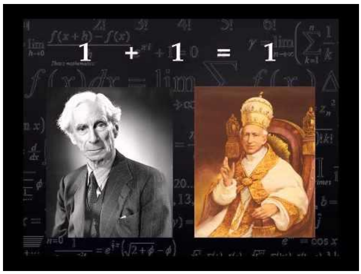
That's enough for these silly slices. Let's move on to the serious business of sets. Now, when set theory was first considered there was no conscious use of axioms. In fact there was only one axiom and it seemed to be so obvious that it didn't seem necessary to state it.

§3.3. Naïve Set Theory

We tend to think of a set a concrete embodiment of some property. Being blue is a property of some physical objects, so we can talk about the set of all blue things. It’s tempting to assume that for every property P there’s a set $\{x \mid Px\}$ whose elements are precisely those elements for which the property holds. But this leads to the **Russell Paradox**.

A contradiction is something that cannot be allowed in mathematics. In ordinary life we somehow live with certain contradictions but in mathematics, if just a single contradiction is allowed, one can prove everything.

Bertrand Russell was once challenged about this claim. “Assuming that 1 + 1 = 1 prove that you’re the Pope,” he was asked. Russell gave an argument along the following lines:



Suppose that 1 + 1 = 1.
Now by definition, 1 + 1 = 2.
Therefore 1 = 2.
The Pope and I are two people.
Therefore the Pope and I are one person.
Therefore I am the Pope!

In the nineteenth and early twentieth centuries mathematicians were concerned with the foundations of the subject, and philosophers were concerned with the nature of truth. They developed mathematics on the basis of set theory. There was basically only one axiom about sets that needed to be used to create this mighty edifice, though it was never stated explicitly.

Axiom of Extensionality: For every property P there is a set that consists of all sets that have that property. In symbols: $\{x \mid Px\}$ is always a set.

The empty set is a set because it's $\{x \mid x \neq x\}$.

$\{a, b\}$ is a set because it's $\{x \mid x = a \text{ or } x = b\}$.

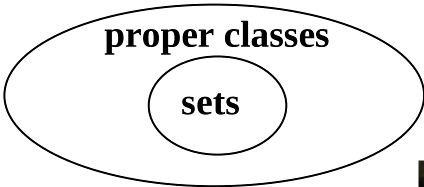
For any set we can define $x^+ = \{x, \{x\}\}$ and hence we can define the natural numbers by defining 0 as the empty set and by considering n^+ as $n + 1$ (though addition and multiplication would yet have to be defined).

In the early 1900s, the great philosopher Frege was preparing the second volume of his book on the foundations of mathematics, building everything on the basis of the axiom of extensionality. But just before it was published Bertrand Russell wrote to him pointing out what we now know as Russell's Paradox.

The contradiction that arises from this paradox shows that the foundation that underpinned Frege's book was invalid. The book had to be withdrawn from publication. Mathematics was in danger of collapsing! A few mathematicians, those interested in the foundations of mathematics, tried to prop it up. Most mathematicians simply ignored the problem and just got on with their own business.

§3.4. Sets and Classes

The rescue came with replacing the one axiom by a set of axioms that avoids the Russell Paradox. We use a class system with ordinary classes called **sets** and elite classes called **proper classes**. So a class is a more general object in that all sets are classes but a class need not be a set.



classes

We define a **set-model** to be a collection of objects, called **sets**, together with a binary relation $\in$ such that:

Axiom of Equality:

$\forall s \forall t[s = t \leftrightarrow \forall x[x \in s \leftrightarrow x \in t]]$

So two sets are equal if and only if they have precisely the same elements.

If $x \in$ S we say that "x is an **element of** S", or "x is a **member of** S", or "S **contains** x". But empty your mind of any intuitive notion you may have of membership.

Everything in set theory is a set. Indeed everything in mathematics can be considered to be a set!

Example 2: Let's take the system ℕ of natural numbers and define membership by: $x \in y \leftrightarrow x^2 < y$.

Then 2 is an 'element' of 9 because $2^2 < 9$. Using the { } notation for listing elements we have 9 = {0, 1, 2} because these are the only natural numbers whose square is less than 9.

But 8 = {0, 1, 2} as well. Since $8 \neq 9$ this violates the axiom of equality. This means that this example is not a set-model.

Example 3: Let's take the collection of natural numbers but this time we'll define $\in$ slightly differently, by:

$$x \in y \leftrightarrow x < y^2.$$

In this model 0 = { } and has no elements since there is no positive integer less than 0^2.
1 = {0} since $0 < 1^2$ but no other numbers.
2 = {0, 1, 2, 3} since $0 < 2^2$ and $1 < 2^2$, $2 < 2^2$ and $3 < 2^2$ but no other numbers.
3 = {0, 1, 2, 3, 4, 5, 6, 7, 8}.

This system *does* satisfy the axiom of equality and so is a set-model. However {0, 1, 2} is a class, but it is not a set because there is no number n such that $x \in n$ if and only if $x = 0$, 1 or 2. Hence {0, 1, 2} is a proper class in this model.

§3.5. Constructions of Classes

I need to say something about the type of predicates that we allow. If we were talking about jelly beans $\{x \mid x$ is red$\}$ is the set of all red jelly beans. But sets can't be red. In fact the only thing they can do is to belong. Proper classes can't even do that. The only predicates that we allow are those that can be built up from the primitive relationship of belonging using the standard logic operations. So $\in$ is a valid predicate and hence so is $\notin$. Equality can be expressed in terms of membership since $x = y$ is equivalent to $\forall z[z \in x \leftrightarrow z \in y]$.

I'll now define some classes that can be constructed from existing classes. The question as to which of them are sets will have to wait.

The **empty class** is $\varnothing = \{x \mid x \neq x\}$.
By the Axiom of Equality, all empty classes are equal, so in any model there is at most one empty set.

The **difference S – T** = $\{x \mid (x \in \mathrm{S}) \wedge (x \notin \mathrm{T}\}$.
The **unordered pair {S, T}** = $\{x \mid (x = \mathrm{S}) \vee (x = \mathrm{T})\}$.
More generally $\boldsymbol{\{x_1, x_2, \ldots, x_n\}}$ denotes:

$$\{x \mid (x = x_1) \vee (x = x_2) \vee \ldots \vee (x = x_n)\}.$$

The **union** of a set S is $\cup\mathbf{S} = \{x \mid \exists y[x \in y \wedge y \in S\}$.

The **intersection** of a non-empty set S is

$$\cap\mathbf{S} = \{x \mid \forall y[y \in S \rightarrow x \in y]\}.$$

You will be familiar with the union and intersection of *two* sets, but $\cup$S and $\cap$S may be something new. But these definitions are just extensions of what you know already. These allow us to talk about the intersection and union of any set of sets.

The **intersection** of two sets S and T is defined to be

$$\mathbf{S \cap T} = \cap\{S, T\}.$$

We say that S, T are **disjoint** if $S \cap T = \varnothing$.

The **union** of two sets S and T is defined to be

$$\mathbf{S \cup T} = \cup\{S, T\}.$$

If S, T are disjoint we often write $S \cup T$ as **S + T**.

Now mathematics contains many more concepts than sets and elements of sets. We have ordered pairs, and integers, and real and complex numbers. There are functions, and matrices and geometric objects such as triangles and circles. Our goal will be to define *all* of these purely in terms of the relation $\in$. In this way we can build up all of mathematics within a certain model. Let's begin with ordered pairs.

How do you define the ordered pair (x, y)? You might say that it consists of two things in a certain order – except that (x, x) is an ordered pair so there might only be one thing. You know intuitively what (x, y) means but giving a precise definition would appear to be tricky. Here's how we do it.

The **ordered pair (x, y)** is defined to be $\{\{x\}, \{x, y\}\}$. This may seem a strange way of defining an ordered pair but it has the one important property that we expect from an ordered pair, namely that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Theorem 6: If $u = (a, b)$ and $v = (c, d)$ then $u = v$ implies that:

$$a = c \text{ and } b = d.$$

Proof: $\cap u = \{a\} \cap \{a, b\} = \{a\}$ and so $\cup\cap u = a$.
Similarly $\cap v = c$. Since $u = v$, we have $a = b$.
Also $\cup u = \{a, b\}$ and so $\cup u - \cap u = \{c\}$.
Hence $\cup(\cup u - \cap u) = b$.
Similarly $\cup(\cup v - \cap v) = d$. Since $u = v$, we have $b = d$. ✋☺

The **cartesian product** S × T is defined to be:

$$\{(x, y) \mid (x \in S) \wedge (y \in T\}$$

S is a **subclass** of T if $\forall x[x \in S \rightarrow x \in T]$.
We denote this by writing **S ⊆ T**.
So S = T if and only if S ⊆ T and T ⊆ S.

S is a **proper subclass** of T if $S \subseteq T$ and $S \neq T$.
We write this as $\mathbf{S} \subset \mathbf{T}$.

The **power class** $\wp(S)$ is defined to be $\{x \mid x \subseteq S\}$.

$\mathbf{S}^+ = \{x \mid (x \in S) \vee (x = S)\}$ called the **successor** of *S*.
In other words $S^+ = S \cup \{S\}$.

A set *x* is called a **successor class** if it contains the successor of each of its elements.

The **successor closure** of S is the intersection of all the successor sets that contain S (as an element). We denote it by **S***. So $S^* = \cap\{x \mid (S \in x) \wedge \forall y(y \in x \rightarrow y^+ \in x)\}$.

Example 3 (continued):

To assist you in understanding all of the above constructions let's see what they are in the model of Example 3. Here the set is ℕ and $x \in y$ means that $x < y^2$.

$\varnothing = 0$ because there are no natural numbers less than 0^2.
$3 - 2$ is the class {4, 5, 6, 7. 8}. Since this is not one of the sets in this model and so it is a proper class.
{2, 3} is not a set in this model and so it is a proper class.
There are no unordered pairs in the model of Example 2.

$\cup 3 = 8$ since $\cup 3 = 0 \cup 1 \cup 2 \ldots \cup 8 = 8$.

$\cap 3 = 0$ since $\cap 3 = 0 \cap 1 \cap \ldots \cap 8 = 0$ since 0 is empty.

$2 \cup 3 = 3$ and $2 \cap 3 = 2$.

$2 \subseteq 3$ since $2 = \{0, 1, 2, 3\}$ and $3 = \{0, 1, 2, \ldots, 8\}$.

$\wp 2 =$ is a proper class
$\wp 3 = 2$ since the only subsets of 3 are 0, 1, 2, 3 and these make up the set 2.

$0^+ = \{0\} = 1$;
$1^+ = 1 \cup \{1\} = \{0\} \cup \{1\} = \{0, 1\}$ which is a proper class;
$2^+ = 2 \cup \{2\} = \{0, 1, 2, 3\} \cup \{2\} = \{0, 1, 2, 3\} = 2$.
In fact $n^+ = n$ for all $n \geq 2$.

One novel feature of this model is that the integers from 2 onwards are elements of themselves. The phenomenon of $x \in x$ is an interesting one. When we come to setting up the axioms for set theory we'll have to decide whether to allow this possibility or whether to rule it out. Before you reach that point in the notes you might like to contemplate whether you would like to allow this self-referential behaviour of sets.

§3.6. ZF-Models

A **ZF-model** is a model that satisfies the following axioms:

(1) Empty Set: $\varnothing$ is a set.
(2) Pairs: If S, T are sets so is {S, T}.
(3) Powers: If S is a set so is $\wp$S.
(4) Union: If S is a set so is $\cup$S.

(5) Infinity: $\omega = \varnothing^*$ is a set.
(6) Specification: If S is a set and P is any predicate built up from $\in$ then $\{x \in S \mid Px\}$ is a set.

(7) Substitution: If S is a set and F is any function then $F[S] = \{F(x) \mid x \in S\}$ is a set.

We'll be defining functions later as sets of ordered pairs. For the purpose of Axiom (7) a **function** is a binary predicate Pxy, built up from $\in$, such that:
$\forall x \forall y \forall z[Pxy \wedge Pxz \rightarrow y = z]$.

Notice that without the Axiom of the Empty Set we'd have no sets at all, because all the other axioms say, "IF S is a set ….". But this axiom on its own only produces one set, $\varnothing$.

By the Axiom of Pairing, if S is any set then so is $\{S, S\} = \{S\}$. So with just the first two axioms we can produce $\varnothing$, $\{\varnothing\}$, $\{\{\varnothing\}\}$, … and pairs of these, such as $\{\{\varnothing\}, \{\{\{\varnothing\}\}\}\}$.

We can produce infinitely many others, such as $\{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$. But all these have 0, 1 or 2 elements.

If we take just the first three axioms we can produce larger sets. For example:
$\wp(\varnothing) = \{\varnothing\}$ with 1 element,

$\wp^2(\varnothing) = \{\varnothing, \{\varnothing\}\}$ with 2 elements,
$\wp^3\varnothing = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}$ with 4 elements.
…………………………………………………..
But all such sets will have size 2^n for some n.

With the first four axioms we can get sets of any finite size. For example:

$$\{\varnothing, \{\varnothing\} \cup \{\varnothing, \{\{\varnothing\}\} = \{\varnothing, \{\varnothing\}, \{\{ \varnothing\}\}\}.$$

But all such sets will be finite.

We need the Axiom of Infinity to get an infinite set and with the Axiom of Specification we can be sure that subclasses of sets are indeed subsets. The Axiom of Substitution is rather more technical than the others, but in essence it says that any class that is in 1-1 correspondence with a set is a set.

But reflect again on the fact that in order for the Axiom of Specification to work we need to have sets in which to operate. And without the Axiom of the Empty Set our model would be empty. The Big Bang that creates the infinite universe of sets from a void is the axiom that assumes the existence of the empty set.

There's something rather appropriate about mathematics being created out of the empty set. If you have a religious bent you can liken it to God creating the world out of nothing. If you have a scientific bent you can liken the process to the Big Bang, which sort of says the same thing, without the religious overtones.

We assume the existence of a ZF-model. This will be our universe of sets within which all of mathematics can be developed. Whether such a model exists is another matter. Of course, things exist in mathematics if we choose to say they exist – *provided they don't lead to a contradiction*.

We could say "let there be a new number, Θ, equal to 0/0". There's nothing wrong with that I suppose, but don't expect the laws of algebra to continue to work like they did when we invented the imaginary number, i.

For if $\Theta = \frac{0}{0}$ then $\Theta + 1 = \frac{0}{0} + \frac{1}{1} = \frac{0.1 + 0.1}{0.1} = \frac{0}{0} = \Theta$,

Similarly Θ + 2 = Θ + 1, so 1 = 2 and you're the Pope!

So the existence of a ZF model hinges purely on whether the ZF axioms are consistent. But they have *never* been proved to be consistent, and probably never will be, because to prove consistency we'd have to create a model that satisfies them, and we can only do this by starting with some sort of model as complex as the ZF model itself. All we can do is to prove theorems based on these axioms and hope for the best!

The ZF axioms are really a creed. Virtually all mathematicians consciously, or unconsciously, believe in this creed, or something equivalent to it. There are a few agnostics who deny the axiom of infinity on the grounds that we live in a finite universe. But they're the losers. Their mathematics is severely impoverished.

Just like a religious creed we can't *prove* that the ZF axioms are true. What would we start with in order to do this?

So here's the point where you can give up mathematics altogether and go and do gardening or something else. If you want to be a serious mathematician and want to base your mathematics on a firm foundation, I'm sorry, the ZF axioms, or their equivalent, are the best we've got.

But, if one day someone comes up with a new paradox that shows the ZF axioms to be inconsistent, a few mathematicians will undertake the job of modifying the fundamental axioms, while the vast majority will continue as if nothing has happened!

Mathematicians have a faith in their mathematical intuition as strong as any religious person does with their religious conviction. Please never say that you must only believe what you can prove!

In the mean time I'll now show that certain familiar mathematical objects that we are familiar with can be considered as sets.

Theorem 7: For all sets S and T, S – T is a set.
Proof: $S - T = \{x \in S \mid x \notin T\}$ is a set by the Axiom of Specification. ✋☺

Theorem 8: $\cap S$ is a set if $S \neq 0$.
Proof: Let $T \in S$.
Then $\cap S = \{x \in T \mid \forall y[y \in S \rightarrow x \in y]\}$ by the Axiom of Specification. ✋☺

Theorem 9: For all sets S and T, $S \cup T$ is a set.
Proof: By the Axiom of Pairs, $\{S, T\}$ is a set.
By the Axiom of Unions, $\cup\{S, T\} = S \cup T$ is a set. ✋☺

Theorem 10: For all sets S, T, (S, T) is a set.
Proof: By the Axiom of Pairs $\{S, T\}$ is a set. By the Axiom of Powers $\wp\{S, T\}$ is a set.
Now $(S, T) = \{\{S\}, \{S, T\}\}$
$= \{x \in \wp(S, T\} \mid x = \{S\} \vee x = \{S, T\}\}$ by the Axiom of Specification. ✋☺

Theorem 11: For all sets S and T, $S \times T$ is a set.
Proof: If $x \in S$ and $y \in T$ then $(x, y) \subseteq \wp\{x, y\}$ and so $(x, y) \in \wp^2\{x, y\}$.
Now each of x and y is an element of $S \cup T$, which is a set by theorem 10.
So $\{x, y\} \subseteq S \cup T$ and so $(x, y) \in \wp^2(S \cup T)$, which is a set by the Axiom of Powers.
Hence $S \times T \subseteq \wp^2(S \cup T)$ and so is a set by the Axiom of Specification. ✋☺

We can write out explicitly the property that extracts $S \times T$ from all the other elements of $\wp^3(S \cup T)$ as follows.

$S \times T =$
$$\{z \in \wp^2(S \cup T) \mid \cup\cap z \in S \text{ and } \cup(\cup z - \cap z) \in T\}.$$

To express the condition in terms of primitive membership statements is straightforward, but very messy. For a start, if $y = \{x \in z \mid Px\}$ is a set we can express
$$y \in S \text{ by } \exists s[s \in S \wedge \forall x[x \in s \leftrightarrow Px]].$$
Now $x \in \cap z$ can be written as $\forall y[y \in z \rightarrow x \in y]$ and $x \in \cup z$ can be written as $\exists y[x \in y \wedge y \in z]$.

So $\cup\cap z \in S$ can be expressed as:
$$\exists s[s \in S \wedge \forall x[x{\in}s \leftrightarrow \exists a[x{\in}a \wedge a{\in}\cap z]]],$$
that is:
$$\exists s[s{\in}S \wedge \forall x[x{\in}s \leftrightarrow \exists a[x{\in}a \wedge \forall b[b{\in}z \rightarrow a{\in}b]]]].$$

And $\cup(\cup z - \cap z) \in T$ can be expressed as:
$$\exists t[t{\in}T \wedge \forall x[x{\in}t \leftrightarrow \exists a[x{\in}a \wedge a \in \cup z - \cap z]]].$$
that is:
$$\exists t[t{\in}T \wedge \forall x[x{\in}t \leftrightarrow \exists a[x{\in}a \wedge \exists b[a{\in}b \wedge b{\in} z] \wedge -\forall b[b{\in}z \rightarrow a{\in}b]]]]$$

I think you get the idea. Now we don't want to have to crawl at this basic level all the way through our development of mathematics. The point is that we *could* do so, if we really had to.

We can't get very far with mathematics without functions and relations. One of the very first things we learnt in arithmetic was 2 + 2 = 4. Before we can justify this rigorously we need not only to define the numbers 2 and 4, but also addition, which is a function of two variables.